THE ASTUMIAN'S PARADOX

Edward W. Piotrowski

Institute of Theoretical Physics, University of Białystok,

Lipowa 41, Pl 15424 Białystok, Poland

e-mail: ep@alpha.uwb.edu.pl

Jan Sładkowski

Institute of Physics, University of Silesia,

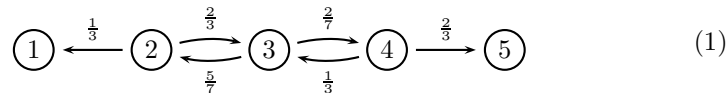Uniwersytecka 4, Pl 40007 Katowice, Poland

e-mail: sladk@us.edu.pl

Abstract

We discuss some aspects of Astumian suggestions that combination of biased games (Parrondo's paradox) can explain performance of molecular motors. Unfortunately the model is flawed by explicit asymmetry overlooked by the author. In addition, we show that taking into consideration stakes allows to remove the paradoxical behaviour of Parrondo games.

Keywords: Parrondo's paradox; random games; martingales; random walk; random transport.

Motto: *Nothing gets out of nothing.* [1]

The celebrated Parrondo's paradox [2, 3] consisting in asymmetrical combination of doomed games so that the resulting new game is not biased or there even is a winning strategy caused much excitement and, unfortunately, misunderstanding. R. D. Astumian [4] considers in a recent article in *Scientific American* a game based on the presented below diagram. The game consist in jumping between five different states $1, \ldots, 5$:

$$
\textcircled{1} \xleftarrow{\frac{1}{3}} \textcircled{2} \underset{\frac{5}{7}}{\overset{\frac{2}{3}}{\rightleftarrows}} \textcircled{3} \underset{\frac{1}{3}}{\overset{\frac{2}{7}}{\rightleftarrows}} \textcircled{4} \xrightarrow{\frac{2}{3}} \textcircled{5} \tag{1}
$$

where the numbers written above or below the arrows are the probabilities of transitions between neighboring states (only such transitions are allowed). The player wins if she (or he) winds up in the state 5 and loses if she reach the state 1. If the player starts from the state 3 the probability of losing is equal to $\frac{5}{9}$ and winning to $\frac{4}{9}$. This is because in games of this kind the proportion of

probabilities of defeat and success is given by the proportion of products of the appropriate transition probabilities

$$\frac{p_3(1)}{p_3(5)} = \frac{p(3 \to 2)\, p(2 \to 1)}{p(3 \to 4)\, p(4 \to 5)} \tag{2}$$

(though it is not true that $p_3(1) = p(3 \to 2)\, p(2 \to 1)$ nor $p_3(5) = p(3 \to 4)\, p(4 \to 5)$). The given above formula has a clear gambling interpretation. Consider the function $M(n)$ (represented in graphic form in Eq. (1)):

Table 1: The martingale corresponding to the game presented in Eq. (1). The states 1 and 5 are absorbing ones.

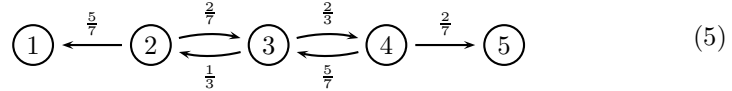| $n =$	1	2	3	4	5
$M(n) =$	$\frac{-1}{p(3\to2)p(2\to1)}$	$\frac{-1}{p(3\to2)}$	0	$\frac{1}{p(3\to4)}$	$\frac{1}{p(3\to4)p(4\to5)}$

$M(n)$ is a martingale [5] that gives capital in a fair game, that is in such stochastic process that at any moment t:

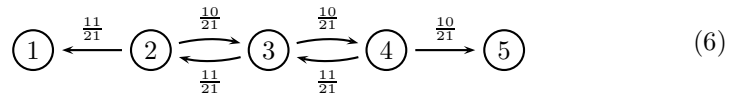$$E(M(n_{t+1})|M(n_0), ..., M(n_t)) = E(M(n_{t+1})|M(n_t)) = M(n_t) \tag{3}$$

(a finite Markov chain). The states 1 and 5 are absorbing, therefore

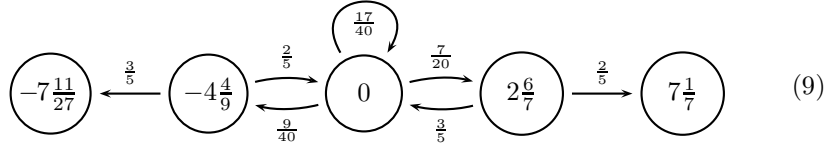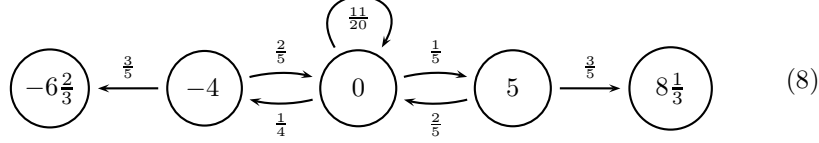$$0 = M(3) = E(M(n_\infty)|n_0{=}3) = p_3(1)M(1) + p_3(5)M(5) \tag{4}$$

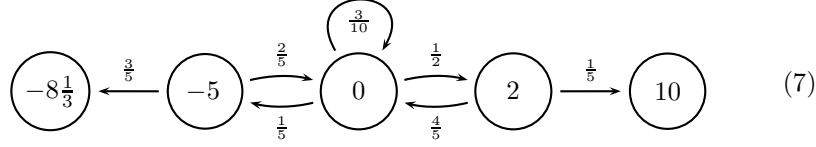and (2) follows. The same numerical result might be obtained in the game

$$\textcircled{1} \xleftarrow{\frac{5}{7}} \textcircled{2} \underset{\frac{1}{3}}{\overset{\frac{2}{7}}{\rightleftharpoons}} \textcircled{3} \underset{\frac{5}{7}}{\overset{\frac{2}{3}}{\rightleftharpoons}} \textcircled{4} \xrightarrow{\frac{2}{7}} \textcircled{5} \tag{5}$$

with different transition probabilities. Astumian suggests that in a modified game with the transition probabilities being arithmetic mean of the probabilities of both above described games that is the game given by

$$\textcircled{1} \xleftarrow{\frac{11}{21}} \textcircled{2} \underset{\frac{11}{21}}{\overset{\frac{10}{21}}{\rightleftharpoons}} \textcircled{3} \underset{\frac{11}{21}}{\overset{\frac{10}{21}}{\rightleftharpoons}} \textcircled{4} \xrightarrow{\frac{10}{21}} \textcircled{5} \tag{6}$$

the probability of success is greater than the probability of defeat. He calls it the player paradox. Even a superficial analysis of the above diagram suggests that this is wrong: the transitions from the states $2, 3, 4$ to the left are more probable than to the right. It seems that an elementary error in counting probabilities resulted in drawing wrong conclusion by prof. Astumian. There is nothing paradoxical if asymmetry put in by hand. Roughly speaking, the Parrondo's paradox consist in the fact that sometimes it is better to play n times *game*1 followed by *game*2 rather than n times *game*1 and then n times *game*2. Below we give an example of a pair of stochastic processes biased towards the left absorbing state and their equally weighted mixed process biased towards the right absorbing state (Parondo's paradox). Consider the following three graphs.

$$\left(-8\tfrac{1}{3}\right)\xleftarrow{\tfrac{3}{5}}\left(-5\right)\underset{\tfrac{1}{5}}{\overset{\tfrac{2}{5}}{\rightleftarrows}}\left(0\right)\circlearrowleft\tfrac{3}{10}\underset{\tfrac{4}{5}}{\overset{\tfrac{1}{2}}{\rightleftarrows}}\left(2\right)\xrightarrow{\tfrac{1}{5}}\left(10\right)\qquad(7)$$

$$\left(-6\tfrac{2}{3}\right)\xleftarrow{\tfrac{3}{5}}\left(-4\right)\underset{\tfrac{1}{4}}{\overset{\tfrac{2}{5}}{\rightleftarrows}}\left(0\right)\circlearrowleft\tfrac{11}{20}\underset{\tfrac{2}{5}}{\overset{\tfrac{1}{5}}{\rightleftarrows}}\left(5\right)\xrightarrow{\tfrac{3}{5}}\left(8\tfrac{1}{3}\right)\qquad(8)$$

$$\left(-7\tfrac{11}{27}\right)\xleftarrow{\tfrac{3}{5}}\left(-4\tfrac{4}{9}\right)\underset{\tfrac{9}{40}}{\overset{\tfrac{2}{5}}{\rightleftarrows}}\left(0\right)\circlearrowleft\tfrac{17}{40}\underset{\tfrac{3}{5}}{\overset{\tfrac{7}{20}}{\rightleftarrows}}\left(2\tfrac{6}{7}\right)\xrightarrow{\tfrac{2}{5}}\left(7\tfrac{1}{7}\right)\qquad(9)$$

The numbers given at the vertices represent values of the martingales and not their arguments. We can identify corresponding vertices of the mixed process if we appropriately change the stakes (in general the stakes are different). The formalism of martingales, besides offering a fast method of finding probabilities of reaching the asymptotic states removes a lot of the mysteries of the paradoxical behavior. In the mixed process the bias can be compensated by modification of stakes. So an increasing probability of winning not necessarily involves increasing (expected) profits.

The authors of this letter suggest the readers to calculate the probabilities of success in games of this kind. An exemplary listing of a mini-program written in the language *Mathematica 5.0* together with results they obtained can be found in Appendix (cf. [6, 7]).

Note that the paradox is present also in quantum games [8, 9].

References

[1] Janosch, *Cholonek oder der liebe Gott aus Lehm*, Georg Bitter, Recklinghausen (1970).

[2] G. P. Harmer and D. Abbott, *Parrondo's paradox*, *Statistical Science* **14** (1999) 206–213.

[3] G. P. Harmer and D. Abbott, *Losing strategies can win by Parrondo's paradox*, *Nature* **402** (1999) 864.

[4] R. D. Astumian, *Making molecules into motors*, *Scientific American* **7** (2001) 56–64.

[5] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. I and II, Wiley & Sons, New York (1971).

[6] http://www.math.temple.edu/˜zeilberg/mamarim/mamarimhtml/parrondo.html.

[7] D. Velleman and S. Wagon, *Parrondo's paradox*, *Mathematica in Education and Research*, **9** (2001) 85–90; http://library.wolfram.com/infocenter/Articles/4073/.

[8] A. P. Flitney and D. Abbott, *Quantum models of Parrondo's games*, *Physica A* **324** (2003) 152–156.

[9] E. W. Piotrowski and J. Sładkowski, *The next stage: quantum game theory*, in Mathematical Physics Research at the Cutting Edge, ed. C. V. Benton, Nova Science Publishers, Inc., New York (2004) 247–268; quant-ph/0308027.

Appendix

The following numerical calculations in *Mathematica 5.0* show the asymptotic behavior of the Astumian's mixed game.

$In[1] :=$ $A = \text{Table}[\text{Which}[\, m == n == 1 \lor m == n == 5, 1,$
$$m == 1 \land n == 2 \lor m == 3 \land n == 4, \tfrac{1}{3},$$
$$m == 3 \land n == 2 \lor m == 5 \land n == 4, \tfrac{2}{3},$$
$$m == 2 \land n == 3, \tfrac{5}{7}, m == 4 \land n == 3, \tfrac{2}{7}, \text{True}, 0\,],$$
$$\{m, 5\}, \{n, 5\}\,];$$
$$B = A/.\{\tfrac{1}{3} \to \tfrac{5}{7}, \tfrac{2}{3} \to \tfrac{2}{7}, \tfrac{5}{7} \to \tfrac{1}{3}, \tfrac{2}{7} \to \tfrac{2}{3}\};$$
$$C = \tfrac{1}{2}\,A + \tfrac{1}{2}\,B;$$
$$finish[A\_] := \text{Rationalize}[\,\text{N}[\,\text{Nest}[\,(\#.\#)\&, A, 12\,].\{0,0,1,0,0\}, 16\,], 10^{-20}]$$
$$\{\text{MatrixForm}[A], \text{MatrixForm}[B], \text{MatrixForm}[C]\}$$
$$\{finish[A], finish[B], finish[C]\}$$

$Out[5] = $
$$\left\{ \begin{pmatrix} 1 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{7} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{5}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{7} & 0 & \frac{5}{7} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{11}{21} & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{21} & 0 & 0 \\ 0 & \frac{10}{21} & 0 & \frac{11}{21} & 0 \\ 0 & 0 & \frac{10}{21} & 0 & 0 \\ 0 & 0 & 0 & \frac{10}{21} & 1 \end{pmatrix} \right\}$$

$Out[6] = $ $\left\{ \{ \tfrac{5}{9}, 0, 0, 0, \tfrac{4}{9} \}, \{ \tfrac{5}{9}, 0, 0, 0, \tfrac{4}{9} \}, \{ \tfrac{121}{221}, 0, 0, 0, \tfrac{100}{221} \} \right\}$